Pre–Calculus 12 Name ________________

7.9 – Natural Logarithms and Related Exponential Functions Block ______

Determine the value of an investment of $1 after 1 year at a rate of 100%/annum for each of the following compound periods:

a) annually $P = 1(2)^1 = \boxed{\$2.00}$

$100+100 = 200\%$

$\to = 1(1+1)^1$

b) semi-annually $P = 1(1.5)^{1/\frac{1}{2}}$

$100 + 50 = 150\%$

$= 1(1.5)^2 = \boxed{\$2.25}$

$\to 1(1+\frac{1}{2})^2$

c) monthly

$P = 1(1+\frac{1}{12})^{12} = \boxed{\$2.61}$

d) bi-weekly

$P = 1(1+\frac{1}{26})^{26} = \boxed{\$2.67}$

e) weekly $P = 1(1+\frac{1}{52})^{52} = \boxed{\$2.69}$

f) daily $P = 1(1+\frac{1}{365})^{365} = \boxed{\$2.71}$

g) hourly $P = 1(1+\frac{1}{8760})^{8760} = \boxed{\$2.72}$

$365 \times 24 = 8760$

h) every minute $P = 1(1+\frac{1}{525600})^{525600} = \boxed{\$2.72}$

$365 \times 24 \times 60 = 525,600$

Notice that as we begin to compound more frequently, the value increases but appears to "level off" at a certain point, specifically a value of $2.72. If money could be compounded infinitely often, we would say it was growing *continuously* and we could find the value using the formula $A = Pe^{rt}$ where P represents the principal, r is the annual interest rate, t is the time in years, and e is the *natural base*. The value of e is approximately 2.71828 and can be calculated using the e^x key on your calculator.

In general, a logarithm with base e is written as $log_e x$. This logarithm is known as the *natural logarithm,* with e being referred to as the *natural base.* However, this is more commonly written simply as lnx (pronounced *"lon x"*).

Definition: $log_e x = lnx$

When $lnx = y$, $e^y = x$ and $x > 0$

Note: $lne = 1$ $(e^1 = e)$

$ln1 = 0$ $(e^0 = 1)$

Money doesn't actually grow continuously, but *population* does, so we can use base e in this context.

Example #1: The formula $P = 2.5e^{(0.052)t}$ can be used to calculate the population P, in millions, of a city with current a population of 2.5 million growing **continuously** at 5.2%/year, where t is the time in years.

a) Determine the population in 10 years. $P = 2.5e^{0.052(10)} = \boxed{4.2 \text{ million}}$

b) Determine when the population will reach 4 million.

$\frac{4}{2.5} = \frac{2.5e^{0.052t}}{2.5}$

$1.6 = e^{0.052t}$

$\ln 1.6 = \ln e^{0.052t}$

$\ln 1.6 = 0.052t \, (\ln e)^{=1}$

$\frac{\ln 1.6}{0.052} = \frac{0.052t}{0.052}$

$\boxed{t = 9.04 \text{ years}}$

Example #2: Use natural logarithms to solve each exponential equation.

a) $3 = e^x$

$\ln 3 = \ln e^x$

$\ln 3 = x \underbrace{\ln e}_{=1}$

$x = \ln 3$

$\boxed{x = 1.1}$

b) $5 = e^{0.03x}$

$\ln 5 = \ln e^{0.03x}$

$\ln 5 = 0.03x \underbrace{\ln e}_{=1}$

$\frac{\ln 5}{0.03} = \frac{\cancel{0.03}x}{\cancel{0.03}}$

$\boxed{x = 53.6}$

c) $\frac{25}{5} = \frac{5e^{3x}}{5}$

$5 = e^{3x}$

$\ln 5 = \ln e^{3x}$

$\ln 5 = 3x \underbrace{\ln e}_{=1}$

$\frac{\ln 5}{3} = \frac{\cancel{3}x}{\cancel{3}}$

$\boxed{x = 0.5}$

Example #3: Solve each logarithmic equation. Write the solution to the nearest thousandth if necessary.

a) $2\ln x + \ln 2 = 4$

$\ln x^2 + \ln 2 = 4$

$\ln \boxed{x^2(2)} = \ln \boxed{e^4}$

$\frac{2x^2}{2} = \frac{e^4}{2}$

$\sqrt{x^2} = \sqrt{\frac{e^4}{2}}$

$x = \sqrt{\frac{e^4}{2}}$

$\boxed{x = 5.2}$

b) $-\ln x + \ln(x-2) = \ln(x+2) - \ln 2x$

$+\ln x$ $\quad$ $+\ln 2x$

$\ln 2x + \ln(x-2) = \ln(x+2) + \ln x$

$\ln \boxed{2x(x-2)} = \ln \boxed{x(x+2)}$

$2x^2 - 4x = x^2 + 2x$

$x^2 - 6x = 0$

$x(x-6) = 0$

$\cancel{x = 0}$ $\quad$ $\boxed{x = 6}$

extraneous root

Homework: p. 487 #1–7 (note: 7c is a challenge)